Derivation of the Ambartsumian's integral equation

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1 Background

In this report, we use a left-handed spherical coordinate system with the z-axis directed opposite to the normal direction of the material's surface(see Figure 1). The light propagation direction is specified by the pair η and φ, where η is the cosine value of the zenith angle and φ is the azimuth angle.

The BRDF of a surface is defined as the ratio between the outgoing radiance and incident irradiance [Nicodemus et al. 1992]:

$$R(\eta, \varphi, \eta_o, \varphi_o) = \frac{L(0, \eta, \varphi, \eta_o, \varphi_o)}{E(\eta_o, \varphi_o)\eta_o} \tag{1}$$

where φ_o and η_o are the azimuth angle and the cosine value of the zenith angle for the incident light respectively. $L(0, \eta, \varphi, \eta_o, \varphi_o)$ is the radiance along direction (η, φ) at the surface(optical depth τ equals to zero), E is the source irradiance. Since the material is composed of randomly distributed particles the scattering and hence the BRDF will be independent of φ_o. So we can decrease one degree of freedom by setting the incident light's azimuth angle φ_o equals to 0, and for simplicity, we will omit φ_o in the rest of the report.

The surfaces of many real-world objects may be considered as layered surfaces, which can be approximated as plane-parallel media [Ishimaru 1997]. A plane-parallel material assumes that the radiance field inside of the material is constant throughout the horizontal plane for given optical depth τ. Under this assumption, the radiative transfer equation can be simplified to the plane-parallel Radiative Transfer Equation(PRTE) [Sobolev 1975]:

$$\begin{aligned} \eta\frac{\mathrm{d}L(\tau, \eta, \varphi, \eta_o)}{\mathrm{d}\tau} =& - L(\tau, \eta, \varphi, \eta_o) + S(\tau, \eta, \varphi, \eta_o) \\ S(\tau, \eta, \varphi, \eta_o) =& \frac{\alpha}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi'\int_{-1}^{1} L(\tau, \eta', \varphi', \eta_o) f_p(\theta')d\eta' \\ &+ \frac{\alpha}{4}E(\eta_o) f_p(\theta) e^{-\tau/\eta_o}, \end{aligned} \tag{2}$$

where L is the radiance due to subsurface scattering, (η,φ) and (η_o,0) are the light propagation direction and the source light direction respectively. $\alpha = \sigma_s/\sigma_t$ is the single scattering albedo, E is the incident irradiance. f_p is the phase function and θ, θ' refer to the scattering angle. $S(\tau, \eta, \varphi, \eta_o)$ is the source function, it represents the aggregate radiance scattering into the direction (η, ξ) which can be separated into two parts: the gain from the scattered light and the gain from the attenuated light source.

We assume that there is no diffuse radiation contribution from either above or below the surface of the material. Then the PRTE should satisfy the following boundary conditions:

$$\begin{aligned} L(0, \eta, \varphi, \eta_o) = 0, \quad \eta > 0 \\ L(\tau_{max}, \eta, \varphi, \eta_o) = 0, \quad \eta < 0 \end{aligned} \tag{3}$$

2 Ambartsumian's integral equation

In this section, we give a brief derivation of the Ambartsumian's integral equaion. The general idea is to decrease the degree of freedom of the PRTE by expanding all the spherical functions into fourier series and then find out the relationship between the reflectance function and the phase function.

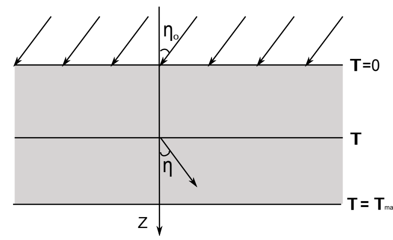

Figure 1: *Configuration used in our derivation*

The first step is to Fourier expand all the terms of PRTE and generate a set of equation with one less degree of freedom. The expansion is carried out as follows:

$$f_p(\theta) = p^0(\eta, \eta') + 2\sum_{m=1}^{n} p^m(\eta, \eta')\cos(m(\varphi - \varphi')) \tag{4}$$

$$S(\tau, \eta, \varphi, \eta_o) = S^0(\tau, \eta, \eta_o) + 2\sum_{m=1}^{n} S^m(\tau, \eta, \eta_o)\cos(m\varphi) \tag{5}$$

$$L(\tau, \eta, \varphi, \eta_o) = L^0(\tau, \eta, \eta_o) + 2\sum_{m=1}^{n} L^m(\tau, \eta, \eta_o)\cos(m\varphi), \tag{6}$$

Substituting these expanded terms into equation 2 we get:

$$\eta\frac{\mathrm{d}L^m(\tau, \eta, \eta_o)}{\mathrm{d}\tau} = -L^m(\tau, \eta, \eta_o) + S^m(\tau, \eta, \eta_0), \tag{7}$$

where

$$\begin{aligned} S^m(\tau, \eta, \eta_o) = &\frac{\alpha}{2}\int_{-1}^{1} p^m(\eta, \eta')L^m(\tau, \eta', \eta_o)d\eta' \\ &+ \frac{\alpha}{4}E(\eta_o)p^m(\eta, \eta_o)e^{-\tau/\eta_o} \end{aligned} \tag{8}$$

and their associated boundary conditions are:

$$\begin{aligned} L^m(0, \eta, \eta_o) = 0, \quad \eta > 0 \\ L^m(\tau_{max}, \eta, \eta_o) = 0, \quad \eta < 0 \end{aligned} \tag{9}$$

The solutions of the ordinary differential equation 7 using these boundary conditions are:

$$L^m(\tau, \eta, \eta_o) = \frac{1}{\eta}\int_0^{\tau} S^m(\tau', \eta, \eta_o)e^{-(\tau-\tau')/\eta}\mathrm{d}\tau', \quad \eta > 0 \tag{10}$$

$$L^m(\tau, \eta, \eta_o) = -\frac{1}{\eta}\int_{\tau}^{\tau_{max}} S^m(\tau', \eta, \eta_o)e^{-(\tau-\tau')/\eta}\mathrm{d}\tau', \quad \eta < 0 \tag{11}$$

We next write the Fourier expansion of the BRDF function:

$$R(\eta, \varphi, \eta_o) = R^0(\eta, \eta_o) + 2\sum_{m=1}^{n} R^m(\eta, \eta_o)\cos(m\varphi) \tag{12}$$

According to the definition of the BRDF in equation 1, we have:

$$R^m(\eta, \eta_o) = \frac{L(0, -\eta, \eta_o)}{E(\eta_o)\eta_o}, \quad \eta > 0 \tag{13}$$

Note that we use the convention that η is positive in R^m which represents the cosine of the angle of emergence w.r.t. the outward normal. And then by setting τ equals to zero and τ_{max} to infinity in equation 11 and combining the result with equation 13 we obtain:

$$R^m(\eta, \eta_o) = \frac{1}{\eta\eta_o E(\eta_o)}\int_0^{\infty} S^m(\tau', -\eta, \eta_o)e^{-\tau'/\eta}\mathrm{d}\tau' \tag{14}$$

The following set of algebraic manipulations will help us simplify the integral term on the right.

Substituting equations 10 and 11 into equation 8 and setting τ_{max} equals to infinity we get:

$$\begin{aligned}
&S^m(\tau,\eta,\eta_o) \\
&= \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,\eta')\mathrm{d}\eta' \int_0^\tau S^m(\tau',\eta',\eta_o)e^{-(\tau-\tau')/\eta'}\mathrm{d}\tau' \\
&+ \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,-\eta')\mathrm{d}\eta' \int_\tau^\infty S^m(\tau',-\eta',\eta_o)e^{-(\tau'-\tau)/\eta'}\mathrm{d}\tau' \\
&+ \frac{\alpha}{4}E(\eta_o)p^m(\eta,\eta_o)e^{-\tau/\eta_o}
\end{aligned} \tag{15}$$

If we set τ to zero, then the first term of the right hand side of equation 15 becomes zero and we obtain:

$$\begin{aligned}
&S^m(0,\eta,\eta_o) \\
&= \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,-\eta') \int_0^\infty S^m(\tau',-\eta',\eta_o)e^{-\tau'/\eta'}\mathrm{d}\tau'\mathrm{d}\eta' \\
&+ \frac{\alpha}{4}E(\eta_o)p^m(\eta,\eta_o)
\end{aligned} \tag{16}$$

Differentiating equation 15 w.r.t. τ, we get:

$$\begin{aligned}
&\frac{\mathrm{d}S^m(\tau,\eta,\eta_o)}{\mathrm{d}\tau} \\
&= \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,\eta')\mathrm{d}\eta' \int_0^\tau \frac{\mathrm{d}S^m(\tau',-\eta',\eta_o)}{\mathrm{d}\tau}e^{-(\tau-\tau')/\eta'}\mathrm{d}\tau' \\
&+ \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,-\eta')\mathrm{d}\eta' \int_\tau^\infty \frac{\mathrm{d}S^m(\tau',-\eta',\eta_o)}{\mathrm{d}\tau}e^{-(\tau'-\tau)/\eta'}\mathrm{d}\tau' \\
&+ \frac{\alpha}{2}\int_0^1 \frac{1}{\eta'}p^m(\eta,\eta')S^m(0,\eta',\eta_o)e^{-\tau/\eta'}\mathrm{d}\eta' \\
&- \frac{\alpha}{4\eta_o}E(\eta_o)p^m(\eta,\eta_o)e^{-\tau/\eta_o}
\end{aligned} \tag{17}$$

By using the superposition principle and operator theory, we find the relationship between S^m and $\mathrm{d}S^m$ from equation 15 and 17:

$$\begin{aligned}
\frac{\mathrm{d}S^m(\tau,\eta,\eta_o)}{\mathrm{d}\tau} &= -\frac{1}{\eta_o}S^m(\tau,\eta,\eta_o) \\
&+ \frac{2}{E(\eta_o)}\int_0^1 \frac{1}{\eta'}S^m(0,\eta',\eta_o)S^m(\tau,\eta,\eta')\mathrm{d}\eta'
\end{aligned} \tag{18}$$

Now using $-\eta$ instead of η in equation 18, multiplying the resulting equation by $e^{-\tau/\eta}$ and then integrating over τ from 0 to infinity, we get:

$$\begin{aligned}
&\frac{1}{\eta}\int_0^\infty S^m(\tau,-\eta,\eta_o)e^{-\tau/\eta}\mathrm{d}\tau + \frac{1}{\eta_o}\int_0^\infty S^m(\tau,-\eta,\eta_o)e^{-\tau/\eta}\mathrm{d}\tau \\
&= S^m(0,-\eta,\eta_o) \\
&+ \frac{2}{E(\eta_o)}\int_0^1 \frac{1}{\eta'}S^m(0,\eta',\eta_o)\int_0^\infty S^m(\tau,-\eta,\eta')e^{-\tau/\eta}\mathrm{d}\tau\mathrm{d}\eta'
\end{aligned} \tag{19}$$

Then substituting equation 14 into equation 19 we get equation 20.

$$\begin{aligned}
(\eta+\eta_o)E(\eta_o)R^m(\eta,\eta_o) &= S^m(0,-\eta,\eta_o) \\
&+ 2\eta\int_0^1 S^m(0,\eta',\eta_o)R^m(\eta,\eta')\mathrm{d}\eta'
\end{aligned} \tag{20}$$

Substituting equation 14 into equation 16, we obtain:

$$\begin{aligned}
S^m(0,\eta,\eta_o) &= \frac{\alpha}{4}E(\eta_o)p^m(\eta,\eta_o) \\
&+ \frac{\alpha}{2}\eta_o E(\eta_o)\int_0^1 p^m(\eta,-\eta')R^m(\eta',\eta_o)\mathrm{d}\eta'
\end{aligned} \tag{21}$$

Finally, substituting equation 21 into equation 20 we get the Ambartsumians integral equation.

$$\begin{aligned}
&R^m(\eta,\eta_o) = \frac{\alpha}{4(\eta+\eta_o)}p^m(-\eta,\eta_o) \\
&+ \frac{\alpha\eta_o}{2(\eta+\eta_o)}\int_0^1 p^m(\eta,\eta')R^m(\eta',\eta_o)\mathrm{d}\eta' \\
&+ \frac{\alpha\eta}{2(\eta+\eta_o)}\int_0^1 p^m(\eta',\eta_o)R^m(\eta,\eta')\mathrm{d}\eta' \\
&+ \frac{\alpha\eta\eta_o}{(\eta+\eta_o)}\int_0^1 R^m(\eta,\eta')\mathrm{d}\eta'\int_0^1 p^m(\eta',-\eta'')R^m(\eta'',\eta_o)\mathrm{d}\eta''
\end{aligned} \tag{22}$$

This last equation is termed Ambartsumian's integral equation which relates Fourier coefficients of the BRDF in an integral equation with single scattering albedo and the Fourier coefficients of the Phase function as the known terms. So these integral equations can be numerically solved to compute the BRDF coefficients and in turn the BRDF.

References

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